Chapter 3: Equations and how to manipulate them
Most mathematical relationships of interest in geology are outgrowths of basic definitions of quantitative relationships between measured quantities.
We manipulate those relationships following basic math rules we learned long ago.
Waltham reviews those basic rules in a general way using various geologic examples. In our discussion today we introduce another geologic example and parallel some of the points made by Waltham.
The geologic example we use as a backdrop for reviewing these old rules is the geologic phenomena of isostacy.

Isostacy
• Length of a degree of latitude
• Mass deficiency in the Andes Mountains
• Everest
• The Archdeacon and the Knight
• Mass deficiency = mass of mountains
• Archimedes - a floating body displaces its own weight of water
• Crust and mantle
Airy’s idea is based on Archimedes principle of hydrostatic equilibrium. Archimedes principle states that a floating body displaces its own weight of water.

Airy applies Archimedes’ principle to the flotation of crustal mountain belts in denser mantle rocks.

**Archimedes Principle**

A floating body displaces its own weight of water.

**Mathematical Statement**

\[
M_{\text{water}} = M_{\text{floating object}}
\]

\[
M_w = M_o
\]

**General Case**

Apply basic principle

\[
M_w = M_o
\]

Apply definition

\[
M_w = \rho_w V_w, \quad M_o = \rho_o V_o
\]

Substitution

\[
\rho_o V_o = \rho_w V_w
\]

**Take a Specific Case**

Let the floating object be an ice cube and then ask yourself -

“What is the volume of the displaced water?”

\[
V_o = V_{\text{ice}}, \quad \rho_o = \rho_{\text{ice}} \text{ and } M_o = M_{\text{water}} = \rho_{\text{ice}} V_{\text{ice}}
\]

Modify notation and substitute for constants

\[
\rho_{\text{ice}} V_{\text{ice}} = 0.9 \frac{\text{g}}{\text{cm}^3} V_{\text{ice}}
\]
\[ M_w = M_{\text{ice}} \quad \rightarrow \quad \rho_w V_w = \rho_{\text{ice}} V_{\text{ice}} \quad \rightarrow \quad V_w = V_{\text{disp}} \quad \Rightarrow \quad \frac{L_w V_w}{\rho_w} \]

Now consider what depth of water is displaced by the ice?

Now if our ice cube has a simple cubical shape to it then the horizontal cross section (length and width) of the ice cube and the displaced water will be the same. Only their height (h and d, respectively) will differ.

Thus

\[ V_w = x y h \]

\[ V_{\text{disp}} = x y r \]

Apply definition

h is the total height of the ice cube and d is the depth it extends below the surface

---

Make our geometry as simple as possible

Income

Ice Cube

---

\[ M_w = M_{\text{ice}} \quad \text{Apply definition} \]

\[ \rho_w x y r = \rho_{\text{ice}} x y h \]

m\text{depth ice extends beneath the surface of the water}
The depth of displaced water since \( \rho_{\text{water}} = 1 \)

How high does the surface of the ice cube rest above the water?

Let \( e \) equal the elevation of the top of the ice cube above the surface of the water.

\[ r = \frac{\rho_{\text{water}}}{\rho_r} \]

Divide both sides of equation by \( \rho_r \)

\[ r = 0.9 \quad \text{(substitution)} \]

Specify mathematical relationship

\[ e = h - r \]

\[ e = h - 0.9h \]

\[ e = 0.1h \]

Distribute Distributive axiom in reverse

Most of us would go through the foregoing manipulations without thinking much about them but those manipulations follow basic rules that we learned long ago.

An underlying rule we have been following is the “Golden Rule” - as Waltham refers to it. That rule is that “whatever you do to an equation, the left and right hand sides must remain equal to each other.

So if we add (multiply subtract ...) something to one side we must do the same to the other side.

The operations of addition, subtraction, multiplication and division follow these basic axioms (which we may have forgotten long ago) - the associative, commutative and distributive axioms.

No matter what kind of math you encounter in geological applications - however simple it may be - you must honor the golden rule and properly apply the basic axioms for manipulating numbers and symbols.

Geological Application

Ice Cubes to Mountain Belts
We can extend the simple concepts of equilibrium operating in a glass of water and ice to large scale geologic problems.

From Ice Cubes and Water to Crust and Mantle

The relationship between surface elevation and depth of mountain root follows the same relationship developed for ice floating in water.

Let's look more carefully at the equation we derived earlier

\[ e = 0.1h \]
Given -

\[ e = h - r \]
\[ r = \frac{\rho_{we}}{\rho_w} \]

.... show that

\[ e = \frac{\rho_{we} - \rho_{wc}}{\rho_w} \]

The constant 0.1 is related to the density contrast

\[ \frac{\rho_{we} - \rho_{wc}}{\rho_w} \]

or ...

\[ e = \frac{\Delta \rho}{\rho_w} \]

Which, in terms of our mountain belt applications becomes

\[ e = \frac{\Delta \rho}{\rho_w} \]

Where \( \rho_m \) represents the density of the mantle and \( \Delta \rho = \rho_m - \rho_c \) (where \( \rho_c \) is the density of the crust.

In a moment we’ll expand our discussions of isostacy into specific geologic applications, but first let’s summarize some of the points Waltham makes for us in Chapter 3.

Waltham notes that the basic skills you learned long ago are critical to your being able to extract useful geological information from basic physical relationships.

Those skills included

1) Rearranging simple equations using the “golden rule” and basic math axioms and
2) Combining and Simplifying (and substitution)

His discussion of “Manipulating Expressions Containing Brackets” is just a close inspection of the distributive axiom in application.
Back to isostasy - The ideas we've been playing around with must have occurred to Airy. You can see the analogy between ice and water in his conceptualization of mountain highlands being compensated by deep mountain roots shown below.

In the diagram below left we have an equilibrium condition. In the diagram below right, we have upset this equilibrium. How deep must the mountain root be to stabilize a mountain with elevation \( e \)?

In the diagram below we refer to the compensation depth. This depth is the depth above which the combined weight of a column of mantle and crust of unit horizontal cross section is constant. Regardless of where you are, the total mass of material overlying the compensation depth will be constant.
If the weight of material above a reference depth is not constant then the crust is not in equilibrium or crustal roots will have to extend below that depth to compensate for the mass excess. The relationship that must hold for the combined weight of crust and mantle above the compensation depth allows us to solve for \( r \) (see below)...

Again we have simplified the equation by assuming that the horizontal cross section of these vertical columns has equal area in all cases, hence the surface areas (xy) cancel out and the mass equivalence relationship reduces to the product of the density and thickness (\( l, d, L \) or \( D \)).

\[
\rho_c l + \rho_m d = \rho_c L + \rho_m D
\]

\[
\rho_c l + \rho_m d = \rho_c (c + l + r) + \rho_m D
\]

Take a few moments and verify that

\[
r = \left( \frac{\rho_c}{\rho_m - \rho_c} \right) c
\]
Let’s take Mount Everest as an example, and determine the root required to compensate for the elevation of this mountain mass above sea level.

Given: \( \rho_c = 2.8 \text{ gm/cm}^3 \), \( \rho_m = 3.35 \text{ gm/cm}^3 \), \( e_E \approx 9 \text{ km} \)

\[
g = \left( \frac{2.8}{3.35 - 2.8} \right) e
\]

\[ r \approx 5.1e \]

Thus Mount Everest must have a root which extends ~ 46 kilometers below the normal thickness of the continent at sea level.

Now let’s look at this problem from a dynamic (changing with time) point of view -

Let’s say that you had continental crust that was in equilibrium and that the average elevation across this crustal block was 0 - its been eroded down to sea-level.

Suppose that through some tectonic process you thickened this crust by 9km. What would be the elevation of the resulting mountain?

Assume the same parameters \( \rho_c = 2.8 \text{ gm/cm}^3 \) and \( \rho_m = 3.35 \text{ gm/cm}^3 \), given in the previous example.

Take a few moments to work through that.

Hint: We have to understand the relationship between the various elements of our model. Note that in the equation that we derived, the \( \rho_c \) terms on the right and left canceled out. So we don’t need to know what the “normal” thickness of the crust \( l \) is. We can also see on the right that the additional 9km thickness of crust must be divided between \( e \) and \( r \), i.e. mountain and root. Thus, in our present example we know that 9km = \( e + r \). Thus \( e = 9 - r \) or \( r = 9 - e \).
We have two relationships to work with. 1) the relationship between $r$ and $e$ and 2) the value of the sum of $r$ and $e$.

\[
\begin{align*}
\rho_r &= \left(\frac{\rho_c}{\rho_m - \rho_c}\right) e \\
\rho_r &= \left(\frac{2.8}{3.35 - 2.8}\right) e \\
r &= 5.1e \\
9 - e &= 5.1e \\
9 &= e(5.1 + 1) \\
9 &= e(6.1) \\
e &= \frac{9}{6.1} \approx 1.48 km
\end{align*}
\]

The importance of Isostacy in geological problems is not restricted to equilibrium processes involving large mountain-belt-scale masses. Isostacy also affects basin evolution because the weight of sediment deposited in a basin disrupts its equilibrium and causes additional subsidence to occur. Consider the following problem.

**In Class Problem:** A 500m deep depression on the earth’s surface fills with sandstone of density 2.2 gm/cm$^3$. Assume that the empty basin is in isostatic equilibrium and that normal crustal thickness in surrounding areas is 20km. Calculate the thickness of sediment that must be deposited in the basin to completely fill it. (Use crustal and mantle densities of 2.8 and 3.3 gm/cm$^3$, respectively.)

**Hint:** Compute the initial thickness of the crust beneath the empty basin and assume that the crustal thickness beneath the basin does not change.
Take Home Problem: A mountain range 4km high is in isostatic equilibrium. (a) During a period of erosion, a 2 km thickness of material is removed from the mountain. When the new isostatic equilibrium is achieved, how high are the mountains? (b) How high would they be if 10 km of material were eroded away? (c) How much material must be eroded to bring the mountains down to sea level? (Use crustal and mantle densities of 2.8 and 3.3 g/cm³.)

A few more comments on Isostacy

The product of density and thickness must remain constant in the Pratt model.

At A 2.9 × 40 = 116
At B \( \rho_C \times 42 = 116 \) \( \rho_C = 2.76 \)
At C \( \rho_C \times 50 = 116 \) \( \rho_C = 2.32 \)
Complete your reading of chapters 3 & 4